

REES ALGEBRAS AND p_g -IDEALS IN A TWO-DIMENSIONAL NORMAL LOCAL DOMAIN

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ABSTRACT. The authors [15] introduced the notion of p_g -ideals for two-dimensional excellent normal local domain over an algebraically closed field in terms of resolution of singularities. In this note, we give several ring-theoretic characterization of p_g -ideals. For instance, an \mathfrak{m} -primary ideal $I \subset A$ is a p_g -ideal if and only if the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

1. INTRODUCTION

In [12], Lipman proved that any integrally closed \mathfrak{m} -primary ideal I in a two-dimensional rational singularity (A, \mathfrak{m}) is stable and normal, that is, $I^2 = QI$ holds for some (every) minimal reduction Q and all power of I are integrally closed. This implies that the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n$ is a Cohen-Macaulay normal domain. Moreover, for any two integrally closed \mathfrak{m} -primary ideals I, J in a two-dimensional rational singularity, one can choose general elements $a \in I$ and $b \in J$ so that $IJ = aJ + bI$. This fact implies that the bigraded Rees algebra $\mathcal{R}(I, J)$ is a Cohen-Macaulay ring. An ideal theory in a two-dimensional rational singularity is established based upon these facts.

In [15], the authors introduced the notion of p_g -ideals for two-dimensional normal local domain (A, \mathfrak{m}) having a resolution of singularities over an algebraically closed field. Let $I \subset A$ be an \mathfrak{m} -primary integrally closed ideal. Then there exist a resolution of singularities $f: X \rightarrow \text{Spec } A$ and an anti-nef cycle Z on X so that $I = H^0(\mathcal{O}_X(-Z))$ and $I\mathcal{O}_X$ is invertible. Then we can show that $\ell_A(H^1(\mathcal{O}_X(-Z))) \leq p_g(A)$, where $p_g(A)$ is the geometric genus of A . If equality holds, then Z is called a p_g -cycle and $I = I_Z$ is called a p_g -ideal. We can consider that the notion of p_g -ideals is a natural extension of integrally closed \mathfrak{m} -primary ideal in a two-dimensional rational singularity. See [15] for more details.

The main purpose of this note is to give several ring-theoretic characterization of p_g -ideals. Namely, we prove the following theorem.

Theorem 1.1 (See Theorems 3.2, 4.1). *Let (A, \mathfrak{m}) be a two-dimensional excellent normal local domain over an algebraically closed field. Let $I \subset A$ be an \mathfrak{m} -primary ideal, and let Q be a minimal reduction of I . Then the following conditions are equivalent:*

Date: November 4, 2015.

2000 Mathematics Subject Classification. Primary 13B22; Secondary 13A30, 14B05.

Key words and phrases. p_g -ideal, Rees algebra, normal Hilbert coefficient, Cohen-Macaulay, rational singularity.

This work was partially supported by JSPS Grant-in-Aid for Scientific Research (C) Grant Numbers, 25400050, 26400053, 26400064.

- (1) I is a p_g -ideal.
- (2) $I^2 = QI$ and $\overline{I^n} = I^n$ for every $n \geq 1$, where \overline{J} denotes the integral closure of an ideal J .
- (3) The Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.
- (4) $\overline{e}_2(I) = 0$.

Let us explain the organization of the paper. In Section 2, we recall the definition and several basic properties for p_g -ideals. For instance, $IJ = aJ + bI$ holds true for any two p_g -ideals I, J and general elements $a \in I, b \in J$. In Section 3, we give a characterization of p_g -ideals in terms of normal Hilbert polynomials. Namely, the vanishing of the second normal Hilbert coefficient of I yields that the ideal is a p_g -ideal. See Theorem 3.2 for more general statement. In Section 4, we give a characterization of p_g -ideals in terms of Rees algebras. Namely, an ideal I is a p_g -ideal if and only if the Rees algebra $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain. Applying this result, one can find some examples of p_g -ideals.

2. BASIC RESULTS

Throughout this paper, let (A, \mathfrak{m}) be an excellent two-dimensional normal local domain and $f: X \rightarrow \text{Spec } A$ a resolution of singularities with exceptional divisor $E := f^{-1}(\mathfrak{m})$ unless otherwise specified. Let $E = \bigcup_{i=1}^r E_i$ be the decomposition into irreducible components of E .

First, we recall the definition of p_g -ideals. An \mathfrak{m} -primary ideal I is said to be *represented on X* if the ideal sheaf $I\mathcal{O}_X$ is invertible and $I = H^0(X, I\mathcal{O}_X)$. If I is represented on X , then there exists an anti-nef cycle Z such that $I\mathcal{O}_X = \mathcal{O}_X(-Z)$; I is also said to be *represented by Z* and write $I = I_Z$. Note that such an ideal I is integrally closed in A .

We say that $\mathcal{O}_X(-Z)$ *has no fixed component* if $H^0(\mathcal{O}_X(-Z)) \neq H^0(\mathcal{O}_X(-Z - E_i))$ for every $E_i \subset E$, i.e., the base locus of the linear system $H^0(\mathcal{O}_X(-Z))$ does not contain any component of E .

We denote by $h^1(\mathcal{O}_X(-Z))$ the length $\ell_A(H^1(\mathcal{O}_X(-Z)))$. It is known that $h^1(\mathcal{O}_X)$ is independent of the choice of the resolution of singularities. The invariant $p_g(A) := h^1(\mathcal{O}_X)$ is called the *geometric genus* of A .

Theorem 2.1 ([15, Theorem 3.1]). *Let $Z > 0$ be a cycle. Suppose that $\mathcal{O}_X(-Z)$ has no fixed component. Then we have the following.*

- (1) $h^1(\mathcal{O}_X(-Z)) \leq p_g(A)$.
- (2) If $h^1(\mathcal{O}_X(-Z)) = p_g(A)$, then $\mathcal{O}_X(-Z)$ is generated (by global sections).

Based upon the theorem above, the authors [15] introduced the notion of p_g -ideals. The definition of p_g -ideal is independent of the choice of the resolution of singularities ([15, Lemma 3.4]).

Definition 2.2 (p_g -ideals, p_g -cycles). A cycle $Z > 0$ is called a p_g -cycle if $\mathcal{O}_X(-Z)$ is generated and $h^1(X, \mathcal{O}_X(-Z)) = p_g(A)$. An \mathfrak{m} -primary ideal I is called a p_g -ideal if I is represented by a p_g -cycle on some resolution.

Assume that $p_g(A) = 0$. Such a ring A is called a rational singularity. Then every (anti-nef) cycle is a p_g -cycle (Lipman [12]).

If $p_g(A) > 0$, then there exists the smallest cycle $C_X > 0$ on X such that $h^1(C_X) = p_g(A)$. The cycle C_X is called the *cohomological cycle* on X . Note that if A is Gorenstein and the resolution of singularities $f: X \rightarrow \text{Spec } A$ is minimal, then $C_X = Z_{K_X}$, where K_X denotes the canonical divisor and Z_{K_X} is the effective cycle such that $K_X = -Z_{K_X}$.

We have the following characterization of p_g -ideals in terms of cohomological cycle.

Proposition 2.3 ([15, Proposition 3.10]). *Assume that $p_g(A) > 0$. Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. Then Z is a p_g -cycle if and only if $\mathcal{O}_{C_X}(-Z) \cong \mathcal{O}_{C_X}$.*

The following theorem plays an important role in this paper.

Theorem 2.4 (Kato's Riemann-Roch formula [11]). *Let I be an \mathfrak{m} -primary ideal such that $I = H^0(\mathcal{O}_X(-Z))$ for some anti-nef cycle Z on X . Then we have*

$$\ell_A(A/I) + h^1(\mathcal{O}_X(-Z)) = -\frac{Z^2 + K_X Z}{2} + p_g(A),$$

where K_X denotes the canonical divisor of X .

In what follows, let us discuss whether $Z + Z'$ is a p_g -cycle. Let Z, Z' be anti-nef cycles on the resolution $X \rightarrow \text{Spec } A$ such that $\mathcal{O}_X(-Z)$ and $\mathcal{O}_X(-Z')$ are generated. Take general elements $a \in I_Z, b \in I_{Z'}$ and put

$$\begin{aligned} \varepsilon(Z, Z') &:= \ell_A(I_{Z+Z'}/aI_{Z'} + bI_Z) \\ &= p_g(A) - h^1(\mathcal{O}_X(-Z)) - h^1(\mathcal{O}_X(-Z')) + h^1(\mathcal{O}_X(-Z - Z')). \end{aligned}$$

Then $0 \leq \varepsilon(Z, Z') \leq p_g(A)$; see [15, Proposition 2.6].

The following proposition gives an important property of p_g -ideals.

Proposition 2.5 (see [15, Theorem 3.5]). *Let Z, Z' be cycles above.*

- (1) *If Z is a p_g -cycle on the resolution X , then $\varepsilon(Z, Z') = 0$ for any Z' . In particular, if $a \in I_Z$ and $b \in I_{Z'}$ are general elements, then*

$$I_{Z+Z'} = aI_{Z'} + bI_Z.$$

- (2) *Assume that Z is a p_g -cycle. Then Z' is a p_g -cycle if and only if so is $Z + Z'$.*
- (3) *If $Z + Z'$ is a p_g -cycle for some cycle Z' , then so is Z .*

Proof. (1),(2) It follows from [15, Theorem 3.5].

- (3) Let $\alpha \in H^0(\mathcal{O}_X(-Z'))$ be a general element. From the exact sequence

$$(2.1) \quad 0 \rightarrow \mathcal{O}_X(-Z) \xrightarrow{\times \alpha} \mathcal{O}_X(-Z - Z') \rightarrow \mathcal{C} \rightarrow 0$$

we obtain $h^1(\mathcal{O}_X(-Z)) \geq h^1(\mathcal{O}_X(-Z - Z')) = p_g(A)$. Hence $h^1(\mathcal{O}_X(-Z)) = p_g(A)$ by Theorem 2.1. \square

Corollary 2.6 ([15, Corollary 3.6]). *Let I, J be \mathfrak{m} -primary integrally closed ideals.*

- (1) *Assume that I is a p_g -ideal. For any \mathfrak{m} -primary integrally closed ideal J and general elements $a \in I, b \in J$, we have*

$$IJ = aJ + bI.$$

- (2) *If I and J are p_g -ideals, then IJ is also a p_g -ideal.*

(3) If IJ is a p_g -ideal, then so are I and J .

The following properties characterize p_g -ideals; see Theorem 4.1.

Corollary 2.7. *Assume that $I = I_Z$ is a p_g -ideal. Then I^n is integrally closed for all $n \geq 1$, $I^2 = QI$, and $I \subset Q : I$.*

3. THE NORMAL HILBERT POLYNOMIALS

Throughout this section, let (A, \mathfrak{m}) be a two-dimensional excellent normal local domain over an algebraically closed field. It is well-known that there exist integers $\bar{e}_0(I)$, $\bar{e}_1(I)$, $\bar{e}_2(I)$ such that

$$\ell_A(A/\overline{I^{n+1}}) = \bar{e}_0(I) \binom{n+2}{2} - \bar{e}_1(I) \binom{n+1}{1} + \bar{e}_2(I) \quad \text{for large enough } n \gg 0.$$

Then

$$P_I(n) = \bar{e}_0(I) \binom{n+2}{2} - \bar{e}_1(I) \binom{n+1}{1} + \bar{e}_2(I)$$

is called the *normal Hilbert polynomial* of I .

Lemma 3.1. *Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. Then:*

- (1) $h^1(\mathcal{O}_X(-nZ)) \geq h^1(\mathcal{O}_X(-(n+1)Z))$ for $n \geq 0$.
- (2) Let

$$n_0 = \min \{n \in \mathbb{Z}_{\geq 0} \mid h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-(n+1)Z))\}.$$

Then $n_0 \leq p_g(A)$ and $h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-n_0Z))$ for $n \geq n_0$.

Proof. (1) follows from the argument of Proposition 2.5.

(2) From the exact sequence

$$0 \rightarrow \mathcal{O}_X(-nZ) \rightarrow \mathcal{O}_X(-(n+1)Z)^{\oplus 2} \rightarrow \mathcal{O}_X(-(n+2)Z) \rightarrow 0,$$

we obtain that $h^1(\mathcal{O}_X(-nZ)) \geq 2h^1(\mathcal{O}_X(-(n+1)Z)) - h^1(\mathcal{O}_X(-(n+2)Z))$. Thus if $h^1(\mathcal{O}_X(-nZ)) = h^1(\mathcal{O}_X(-(n+1)Z))$ is satisfied, then $h^1(\mathcal{O}_X(-(n+1)Z)) = h^1(\mathcal{O}_X(-(n+2)Z))$ holds true. \square

By using Riemann-Roch formula, we describe the normal Hilbert-polynomial of $I = I_Z$ in terms of the cycle Z .

Theorem 3.2. *Assume that I is represented by a cycle $Z > 0$. Let $P_I(n)$ be a normal Hilbert-polynomial of I . Then*

- (1) $P_I(n) = \ell_A(A/\overline{I^{n+1}})$ for all $n \geq p_g(A) - 1$.
- (2) $\bar{e}_0(I) = e_0(I) = -Z^2$.
- (3) $\bar{e}_1(I) = e_0(I) - \ell_A(A/I) + (p_g(A) - h^1(\mathcal{O}_X(-Z))) = \frac{-Z^2 + ZK_X}{2}$.
- (4) $\bar{e}_2(I) = p_g(A) - h^1(\mathcal{O}_X(-nZ))$ for all $n \geq p_g(A)$.

Proof. It follows from the Riemann-Roch formula that

$$\begin{aligned}\ell_A(A/\overline{I^{n+1}}) &= -\frac{(n+1)^2 Z^2 + (n+1)ZK_X}{2} + p_g(A) - h^1(\mathcal{O}_X(-(n+1)Z)) \\ &= -Z^2 \binom{n+2}{2} - \frac{-Z^2 + ZK_X}{2} \binom{n+1}{1} + p_g(A) - h^1(\mathcal{O}_X(-(n+1)Z)).\end{aligned}$$

Since $h^1(\mathcal{O}_X(-nZ))$ is stable for $n \geq p_g(A)$ by Lemma 3.1, we obtain the required assertions. \square

As a corollary, we obtain a simple characterization of p_g -ideals in terms of normal Hilbert coefficients.

Corollary 3.3. *The following conditions are equivalent:*

- (1) I is a p_g -ideal.
- (2) $\bar{e}_1(I) = e_0(I) - \ell_A(A/I)$.
- (3) $\bar{e}_2(I) = 0$.

Proof. (1) \implies (2) follows from the theorem above.

(2) \implies (3) : By assumption, $I = I_Z$ is a p_g -ideal. Hence $I_{nZ} = I^n$ is a p_g -ideal by Corollary 2.6(2), and thus $\bar{e}_2(I) = 0$ by the theorem above.

(3) \implies (1) : The theorem above yields that $h^1(\mathcal{O}_X(-(n+1)Z)) = p_g(A)$ for $n \gg 0$ and thus $I_{nZ} = I^n$ is a p_g -ideal. By Corollary 2.6(3), we obtain that I_Z is also a p_g -ideal. \square

For any cycle Z on X , we put $Z^\perp = \sum_{ZE_i=0} E_i$.

Proposition 3.4. *Let $Z > 0$ be a cycle such that $\mathcal{O}_X(-Z)$ has no fixed component. If C is the cohomological cycle on Z^\perp , i.e., the smallest cycle with*

$$h^1(\mathcal{O}_C) = \max_{D>0, D_{\text{red}} \leq Z^\perp} h^1(\mathcal{O}_D),$$

then $\mathcal{O}_C \cong \mathcal{O}_C(-n_0 Z)$ and $h^1(\mathcal{O}_C) = h^1(\mathcal{O}_X(-n_0 Z)) = p_g(A) - \bar{e}_2(I_Z)$, where n_0 is an integer given by Lemma 3.1.

Proof. Let $D > 0$ satisfy that $\text{Supp}(D) = Z^\perp$ and $DE_i < 0$ for all $E_i \leq Z^\perp$. There exist $m, n \in \mathbb{N}$ such that $H^1(\mathcal{O}_X(-nD - mZ)) = 0$ (cf. the proof of [15, Proposition 3.10]). Then $H^1(\mathcal{O}_X(-mZ)) = H^1(\mathcal{O}_{nD}(-mZ))$. Since $\mathcal{O}_{nD}(-mZ) \cong \mathcal{O}_{nD}$, $h^1(\mathcal{O}_X(-mZ)) = h^1(\mathcal{O}_C)$ for sufficiently large m . \square

Remark 3.5. Assume that $\mathcal{O}_X(-Z)$ is generated. Let $E^{(1)}, \dots, E^{(k)}$ be the connected components of Z^\perp and assume that each $E^{(i)}$ contracts to a normal surface singularity isomorphic to (A_i, \mathfrak{m}_i) . Then we have $p_g(A) = \bar{e}_2(I_Z) + \sum_{i=1}^k p_g(A_i)$ (cf. [14, Corollary 4.5]).

Example 3.6. Let $e \geq 2$ be an integer, and let $A = k[[x, y, z]]/(x^e + y^e + z^e)$. Then the Poincare series of $k[x, y, z]/(x^e + y^e + z^e)$ is equal to

$$\sum_{k \geq 0} \ell_A(\mathfrak{m}^k / \mathfrak{m}^{k+1}) t^k = \frac{1 - t^e}{(1 - t)^3} = \frac{1 + t + t^2 + \dots + t^{e-1}}{(1 - t)^2}.$$

It follows that

$$\ell_A(A/\mathfrak{m}^{n+1}) = \begin{cases} e \binom{n+2}{2} - \frac{e(e-1)}{2} \binom{n+1}{1} + \frac{e(e-1)(e-2)}{6} & (n \geq e), \\ \frac{(n+1)(n+2)(n+3)}{6} & (n \leq e-1). \end{cases}$$

Hence

$$\begin{cases} \bar{e}_0(\mathfrak{m}) &= e_0(\mathfrak{m}) = e, \\ \bar{e}_1(\mathfrak{m}) &= e_1(\mathfrak{m}) = \frac{e(e-1)}{2}, \\ \bar{e}_2(\mathfrak{m}) &= e_2(\mathfrak{m}) = \frac{e(e-1)(e-2)}{6} = p_g(A); \quad \text{see [18, (4.11)]}. \end{cases}$$

In particular, $h^1(\mathcal{O}_X(-kZ)) = 0$ for every $k \geq e$.

On the other hand,

$$\frac{e(e-1)}{2} = \bar{e}_1(\mathfrak{m}) = e_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) + p_g(A) - h^1(\mathcal{O}_X(-Z))$$

$$\text{yields } h^1(\mathcal{O}_X(-Z)) = \frac{(e-1)(e-2)(e-3)}{6}.$$

Furthermore, since $ZK = 2 \cdot \bar{e}_1(\mathfrak{m}) - (-Z^2) = e(e-2)$ and $e_0(\mathfrak{m}^k) = k^2e$, we have

$$\begin{aligned} h^1(\mathcal{O}_X(-kZ)) &= e_0(\mathfrak{m}^k) - \ell_A(A/\mathfrak{m}^k) + p_g(A) - \frac{-(kZ)^2 + (kZ)K}{2} \\ &= k^2e - \binom{k+2}{3} + \binom{e}{3} - \frac{k^2e + ke(e-2)}{2} \\ &= \frac{(e-k)(e-k-1)(e-k-2)}{6} = \binom{e-k}{3} \end{aligned}$$

for each $k = 1, 2, \dots, e-1$. In particular, we get

$$h^1(\mathcal{O}_X(-(e-3)Z)) = 1 \quad \text{and} \quad h^1(\mathcal{O}_X(-(e-2)Z)) = 0,$$

and thus $n_0 = e-2$ in Lemma 3.1.

4. THE REES ALGEBRA

Let (A, \mathfrak{m}) be a Cohen-Macaulay local ring of dimension d , and let I be an ideal of A . Now consider three A -algebras, which are called blow-up algebras.

$$\mathcal{R}(I) := A[It] = \bigoplus_{n \geq 0} I^n t^n \subset A[t].$$

$$\mathcal{R}'(I) := A[It, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n \subset A[t, t^{-1}].$$

$$G(I) := \mathcal{R}(I)/I\mathcal{R}(I) \cong \mathcal{R}'(I)/t^{-1}\mathcal{R}'(I).$$

The algebra $\mathcal{R}(I)$ (resp. $\mathcal{R}'(I)$, $G(I)$) is called the *Rees algebra* (resp. *the extended Rees algebra*, *the associated graded ring*) of I .

The main purpose of this section is to characterize p_g -ideals in terms of blow-up algebras.

Theorem 4.1. *Let (A, \mathfrak{m}) be a two-dimensional excellent normal local domain over an algebraically closed field, and let $I \subset A$ be an \mathfrak{m} -primary ideal. Then the following conditions are equivalent:*

- (1) *I is a p_g -ideal in the sense of Definition 2.2.*
- (2) *$I^2 = QI$ for some minimal reduction Q of I , and $\overline{I^n} = I^n$ holds true for every $n \geq 1$.*
- (3) *$\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.*
- (4) *$\mathcal{R}'(I)$ is a Cohen-Macaulay normal domain with $a(G(I)) < 0$.*

Proof. (1) \implies (2) : It follows from Corollary 2.7.

(2) \implies (3) : Since $I^2 = QI$ for some minimal reduction Q of I , $\mathcal{R}(I)$ is Cohen-Macaulay by Valabrega–Valla [19] and Goto–Shimoda [2]. Moreover, since A is normal and $\overline{I^n} = I^n$ for every $n \geq 1$, $\mathcal{R}(I)$ is a normal domain.

(4) \iff (3) \implies (2) follows from Goto–Shimoda [2] and Herzog et.al [6, Proposition 2.1.2].

(2) \implies (1) : Assume that I^n is integrally closed for $n \geq 1$ and that $I^2 = QI$ for a minimal reduction Q of I . Suppose that I is represented by a cycle Z on X . Consider the following exact sequence given by general elements of $I = I_Z$ and I_{nZ} (see [15, (2.3)]).

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(-Z) \oplus \mathcal{O}_X(-nZ) \rightarrow \mathcal{O}_X(-(n+1)Z) \rightarrow 0.$$

Since $QI^n = I^{n+1} = \overline{I^{n+1}}$, we obtain that $\varepsilon(Z, nZ) = 0$ for $n \geq 1$. Therefore, $p_g(A) = h^1(\mathcal{O}_X(-Z))$ because $h^1(\mathcal{O}_X(-nZ))$ is stable for $n \gg 0$. \square

The following two examples are known.

Example 4.2 (cf. Lipman [12]). Let A be a two-dimensional rational singularity. Then any integrally closed \mathfrak{m} -primary ideal I is a p_g -ideal and $\mathcal{R}(I)$ is a Cohen-Macaulay normal domain.

Example 4.3. Let A be a complete Gorenstein local ring with $p_g(A) > 0$. If \mathfrak{m} is a p_g -ideal of A , then \mathfrak{m} is stable, that is, $\mathfrak{m}^2 = Q\mathfrak{m}$ for some minimal reduction Q of \mathfrak{m} . Since A is Gorenstein, we obtain that A is a hypersurface of degree 2. So we may assume that $A = K[[x, y, z]]/(f)$, where $f = x^2 + g(y, z)$. As A is not rational, $g(y, z) \in (y, z)^3$. Moreover, since $R(\mathfrak{m})$ is normal, we have $g(y, z) \notin (y, z)^4$.

Conversely, if $A = K[[x, y, z]]/(x^2 + g(y, z))$, where $g(y, z) \in (y, z)^3 \setminus (y, z)^4$, then for every n , $\mathfrak{m}^n = (y, z)^n + x(y, z)^{n-1}$ and is integrally closed. Then, since \mathfrak{m} is stable and \mathfrak{m}^n is integrally closed for every $n \geq 1$, \mathfrak{m} is a p_g -ideal.

The next example gives a hypersurface local ring A whose maximal ideal is a p_g -ideal and $p_g(A) = p$ for a given integer $p \geq 1$.

Example 4.4. Let $p \geq 1$ be an integer, and let k be an algebraically closed field. Let $B = k[x, y, z]/(x^2 + y^3 + z^{6p+1})$. If we put $\deg x = 3(6p+1)$, $\deg y = 2(6p+1)$

and $\deg z = 6$, then A can be regarded as a quasi-homogeneous k -algebra with $a(A) = 6p - 5$. In particular,

$$p_g(B) = \sum_{i=0}^{6p-5} \dim_k B_i = p; \quad (\text{cf. [16, 23]}).$$

Moreover, if we put $X = xt$, $Y = yt$, $Z = zt$ and $U = t^{-1}$, then the extended Rees algebra of $\mathfrak{m} = (x, y, z)$ is

$$\mathcal{R}'(\mathfrak{m}) \cong k[X, Y, Z, U]/(F),$$

where $F = X^2 + Y^3U + Z^{6p+1}U^{6p-1}$. Since the Jacobian ideal is

$$\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}, \frac{\partial F}{\partial U}, F \right) = (X, Y^2U, Z^{6p}U^{6p-1}, Y^3 + (6p-1)Z^{6p+1}U^{6p-2}),$$

one can check (R_1) -condition of $\mathcal{R}'(\mathfrak{m})$. Thus $\mathcal{R}'(\mathfrak{m})$ is a normal domain because it is Cohen-Macaulay.

Now let us put $A = B_{(x,y,z)}$ and $\mathfrak{m} = (x, y, z)A$. Then we can conclude that A is a two-dimensional normal hypersurface with $p_g(A) = p$ and that \mathfrak{m} is a p_g -ideal by applying the theorem above.

Similarly, if we consider $I_k = (x, y, z^k)A$ and $Q_k = (y, z^k)$ for $k = 2, 3, \dots, 3p$, then $I_k^2 = Q_k I_k$ and $\mathcal{R}'(I_k)$ is a normal domain. Hence I_k is a p_g -ideal.

The next example gives a hypersurface local ring A whose maximal ideal is not a p_g -ideal and $p_g(A) = p$ for a given integer $p \geq 1$.

Example 4.5. Let $p \geq 1$ be an integer. Let $A = k[x, y, z]_{(x,y,z)}/(x^2 + y^4 + z^{4p+1})$. Then A is a two-dimensional normal hypersurface with $p_g(A) = p$. Then $\mathfrak{m} = (x, y, z)$ is *not* a p_g -ideal and $I_k = (x, y, z^k)$ is a p_g -ideal for every $k = 2, 3, \dots, 2p$ because $\mathcal{R}'(I_k)$ is normal but $\mathcal{R}'(\mathfrak{m})$ is not.

Furthermore, \mathfrak{m}^k is not a p_g -ideal for every $k \geq 1$ by Corollary 2.6.

It is not so difficult to extend our result to the case of bigraded Rees algebras. Let $I, J \subset A$ be ideals. Then

$$\mathcal{R}(I, J) := A[It_1, Jt_2] = \bigoplus_{n=1}^{\infty} \bigoplus_{m=1}^{\infty} I^m J^m t_1^m t_2^n \subset A[t_1, t_2]$$

is called the *multi-Rees algebra* of I and J .

Corollary 4.6. *Let (A, \mathfrak{m}) be a two-dimensional excellent normal local domain over an algebraically closed field, and let I, J be \mathfrak{m} -primary ideals. Then the following conditions are equivalent:*

- (1) I and J are p_g -ideals.
- (2) $\mathcal{R}(I, J)$ is a Cohen-Macaulay normal domain.
- (3) I, J are integrally closed and $\mathcal{R}(IJ)$ is a Cohen-Macaulay normal domain.

Proof. (1) \implies (2) : Since I and J are p_g -ideals, $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are Cohen-Macaulay and $IJ = aJ + bI$ for some joint reduction (a, b) of (I, J) . Hence $\mathcal{R}(I, J)$ is Cohen-Macaulay by [8, Corollary 3.5] (see also e.g. [5], [20], [21]). Since $S = \mathcal{R}(I)$ is a normal domain and $J I^k$ is integrally closed for every $k \geq 1$, $\mathcal{R}(I, J)$ is normal.

(2) \implies (1) : Since $\mathcal{R}(I, J)$ is Cohen-Macaulay, $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are Cohen-Macaulay by [8, Corollary 3.5]. Since $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are pure subrings of $\mathcal{R}(I, J)$, they are normal domains. Hence I and J are p_g -ideals by Theorem 4.1.

(1) \iff (3) : It follows from Theorem 4.1 and Corollary 2.6. \square

Remark 4.7. By a similar argument as in the proof of (1) \implies (2), we can obtain that the multi-Rees algebra $\mathcal{R}(I_1, \dots, I_r)$ is a Cohen-Macaulay normal domain for every p_g -ideals I_1, \dots, I_r .

Remark 4.8. Assume that A is a rational singularity. Let I and J be \mathfrak{m} -primary integrally closed ideals of A . Then I and J are p_g -ideals and thus $\mathcal{R}(I)$, $\mathcal{R}(J)$ and $\mathcal{R}(I, J)$ are Cohen-Macaulay normal domains. In fact, S. Goto, N. Taniguchi and the third author [3] prove that $\mathcal{R}(I)$ and $\mathcal{R}(J)$ are almost Gorenstein. Moreover, Verma [21] proved that they admit minimal multiplicities.

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